HSC PHYSICS

Module 5: Advanced Mechanics

Week 4

Motion in Gravitational Fields I

WORKBOOK

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Contents

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Syllabus Content

Motion in Gravitational Fields

- Apply qualitatively and quantitatively Newton's Law of Universal Gravitation to:
  - determine the force of gravity between two objects $F = \frac{GMm}{r^2}$
  - investigate the factors that affect the gravitational field strength $g = \frac{GM}{r^2}$
  - predict the gravitational field strength at any point in a gravitational field, including at the surface of a planet
- Investigate the orbital motion of planets and artificial satellites when applying the relationships between the following quantities:
  - gravitational force, centripetal force, centripetal acceleration, mass, orbital radius, orbital velocity, orbital period
- Investigate the relationship of Kepler's Laws of Planetary Motion to the forces acting on, and the total energy of, planets in circular and non-circular orbits using:
  - $v = \frac{2\pi r}{T}$
  - $\frac{r^3}{T^2} = \frac{GM}{4\pi^2}$

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Quiz

Question 1 (7 marks)

A racing driver navigates a circular part of the flat track at 190 $\mathrm{km\,h^{-1}}$. He turns his wheels 30° to generate the force required for the turn. The driver and the car together weigh 880 kg.

(a) If the radius of the turn is 850 m, calculate the magnitude of inward friction force supplied by the car. **1**

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(b) Calculate the magnitude of total friction force on the tyres. **1**

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(c) Calculate the car's angular frequency and period. **2**

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(d) Explain why, when a tight turn is traversed too fast, the car risks skidding to the outside of the track. In your answer, make reference to different types of friction and inertia. **3**

Question 2 (3 marks)

Calculate the range of a projectile that is thrown from a height of 20 m and lands at ground level, having reached a maximum height of 32 m above the ground and having started with equal vertical and horizontal components of initial velocity.

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1 Newton's Law of Universal Gravitation

Sir Isaac Newton (1642–1727) postulated that all matter in the universe (anything with mass) attracts all other matter in the universe with a force. One observable manifestation of this is the fact that objects fall down towards the surface of the Earth, since the Earth attracts everything on it towards its centre of mass. But by considering the various gravitational interactions between bodies of varying size in the universe, and building off the work of astronomers before him, Newton reasoned that *every* object should be attracting *every other* object. This includes two humans, two atoms, and every single pair of objects you can see before you right now. The only reason those objects don't fall into each other is that the gravitational force between them is absolutely minuscule.

Say, then, that we have two objects, A and B. Just as object A attracts object B with a gravitational force, object B also attracts object A with an equal and opposite force. In this way, gravitation satisfies Newton's Third Law of motion as well. The implication of this is that just as the Earth is pulling you down to the ground, you are also pulling the Earth up towards you with your gravitational pull – with a force of the exact same magnitude, no less!

Newton reasoned that the more massive the two objects in question (i.e. the more matter they contained), the greater would be the force of attraction between them. Mathematically, if m_1 and m_2 are the two objects' masses in kg, then:

$$F \propto m_1 \quad \text{and} \quad F \propto m_2$$

Furthermore, the closer the objects are to each other, the stronger the force of attraction, according to an inverse square relationship. Mathematically, if r is the separation between the object's centres of mass in m, then:

$$F \propto \frac{1}{r^2}$$

By incorporating a constant of proportionality, the gravitational constant G, we can formulate **Newton's law of universal gravitation** (Figure 1):

$$F = \frac{Gm_1m_2}{r^2}$$

where

$$\begin{aligned} m_1, m_2 = &\ \text{masses of two objects (kg)}, \\ r = &\ \text{distance between their centres of mass (m)}, \\ \text{and } G = 6.67x10^{-11}, &\ \text{the gravitational constant (Nm}^2/\text{kg}^2). \end{aligned}$$

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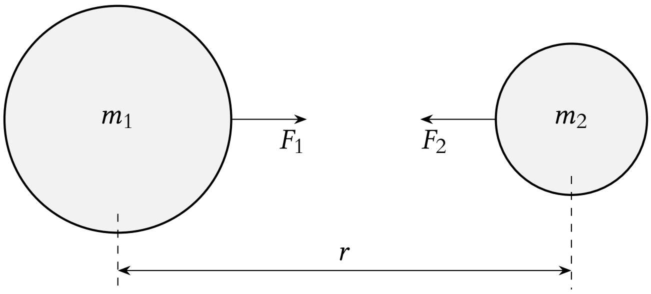

$$F_1 = F_2 = \frac{Gm_1m_2}{r^2}$$

Figure 1 Newton's Law of Universal Gravitation.

The gravitational force is always a force of attraction. When describing the direction of the force between two objects, you can use the word 'attractive' or the phrases 'towards the other object' or 'towards each other' all as appropriate designations of direction – for example, 'F = 18.5 N attractive' or 'F = 18.5 N towards each other'.

Centre of Mass

It is important to remember that r is the distance separating the objects' centres of mass, with this latter term referring to the *single point* inside the object that all its mass is assumed to be concentrated into.

For most questions, the distance between the two objects given to you can be assumed to be between their centres of mass. For example, if you are asked to find the gravitational force between a water bottle and a pen, and you are told they are 2 m apart, you do not need to account for the size of each object in trying to find exactly where their centres are and add that to the 2 m separation. Just assume their centres of mass are 2 m apart.

Similarly, if you are told that the Sun and Earth are 1.496×10^{11} m apart, assume that refers to their centres of mass unless told otherwise.

The exception to this is when an object is at or reasonably close to the surface of a large celestial body. When asked to find the gravitational attraction on a person standing on the surface of the Earth, the separation between the person and the Earth *is not* zero, it is the distance to the Earth's centre.

Using common sense, you should generally be able to work out quite easily in a given question if you need to account for the shape of the object or not in establishing the value of r.

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The Gravitational Constant

The gravitational constant G is the constant of proportionality in Newton's law of universal gravitation. How do we know that it is equal to 6.67×10^{-11}?

This is a value that has been experimentally derived and refined over many years. In the late 1700s, Henry Cavendish conducted an experiment to determine the tiny deflection caused to a suspended object when placed next to a larger fixed object as they attract each other by the gravitational force. Modern experimental data allows us to be certain of the value to 5 decimal places, which – surprisingly – isn't that high a certainty in comparison to our knowledge of other physical constants of the universe.

Weight and Gravitational Acceleration

Previously, we have been using a different formula to determine the force on an object due to gravity, which is the weight formula $W = mg$. Although this formula works well for objects on the surface of a planet, and when we already know the value for g, it really doesn't help us to determine the broader set of gravitational attractions which occur in the universe – for example the force between two small objects, or the force between the Earth and the Sun. Furthermore, if we don't know the value of g on a particular planet, how do we calculate the gravitational force between that planet and objects on its surface?

This is where Newton's law of universal gravitation is very powerful. We can pick any two objects we are interested in, and as long as we know their masses and separation, we can find the force of gravity that exists between the two.

We can use the universal gravitation formula in combination with the weight formula (which is just Newton's Second Law) to derive a formula for the gravitational acceleration of a celestial body.

Consider the gravitational force acting on a small object, such as a person, by a much larger body, such as the Earth. Instead of referring to their masses as m_1 and m_2, let's instead call the small mass m and the big mass M. Then the universal gravitational formula looks as follows:

$$F = \frac{GmM}{r^2}$$

This is a common way to see Newton's law of universal gravitation written down when the two objects in consideration are orders of magnitude different in mass. It is the same formula, just written in such a way as to remind us of the physical situation we are dealing with.

In the weight formula $W = mg$, the weight is the gravitational force on the small object (and, by Newton's Third Law, also the large object but in the opposite direction), m is the mass of the small object, and g is the acceleration due to gravity due to the large object.

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Equating the two, we get:

$$W = F$$
$$mg = \frac{GmM}{r^2}$$
$$g = \frac{GM}{r^2}$$

This formula allows us to calculate in $\mathrm{m\,s^{-2}}$ the gravitational acceleration of a celestial body at any distance away from its centre of mass. We can see that the heavier the celestial body is, the stronger its gravitational acceleration is. Also, the further you are away from its centre of mass, the weaker that acceleration will be. Note that it is independent of the mass of the small object experiencing that acceleration, exactly as Galileo predicted in his (supposed) Leaning Tower of Pisa experiments.

Advanced: Gravitational and Inertial Mass

In the derivation above, we cancelled out m from each side, saying that they are the same term representing the mass of the small object. Although this is widely accepted as obviously correct, there is no logical reason from first principles why this should be the case.

The m in $W = mg$ refers to **inertial mass**. It is the property of an object that links a force applied on it to the resultant acceleration, i.e. $F = ma$. The higher this number is, the lower the resultant acceleration caused by a particular force.

The m in $\frac{GmM}{r^2}$ refers to **gravitational mass**. It is the property of an object that results in it exerting a gravitational force on every other object with gravitational mass, just like charge is the property of an object that results in it exerting an electrostatic force on every other charged object.

Why should the property of an object which causes its gravity necessarily be the same property which determines what acceleration it will undergo when it encounters any force? Just because we have referred to them both as mass, *are* they both the same thing?

For centuries, scientists were confident without definitive proof that they were probably the same. We can show experimentally that objects of different masses will accelerate down at similar rates when falling under the influence of Earth's gravity. This idea supports our above derivation, because if we could not cancel out the two masses, there would be a dependence on mass left over in the formula for gravitational acceleration. But just because experiments support the idea of the two masses being the same thing, it doesn't provide us with the underlying reason *why*.

It took several hundred years and an innovative thought experiment by Albert Einstein – yes, an experiment conducted in his head – to actually *prove* that the gravitational and inertial masses are identical. This is known as the **equivalence**

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principle and forms the basis of the theory of general relativity, which we will not be covering in this course.

Questions

Question 1 (2 marks)

A water bottle of mass 200 g and a dumbbell of mass 3 kg are placed on the floor of a gym, 2 m apart. What is the gravitational force between these objects?

Solution to Question 1

$$\begin{aligned} F &= \frac{GmM}{r^2} \\ &= \frac{6.67 \times 10^{-11} \times 0.2 \times 3}{2^2} \\ &= 1 \times 10^{-11} \text{ N attractive} \end{aligned}$$

Marking Criteria

- Correctly substitutes values into formula (1 mark) and correctly calculates answer (1 mark)

Question 2 (2 marks)

The Sun weighs 1.99×10^{30} kg while the planet Mars weighs 6.42×10^{23} kg. If the gravitational force between them is 1.64×10^{21} N, calculate the distance of Mars from the Sun.

Solution to Question 2

$$\begin{aligned} F &= \frac{GmM}{r^2} \\ 1.64 \times 10^{21} &= \frac{6.67 \times 10^{-11} \times 1.99 \times 10^{30} \times 6.42 \times 10^{23}}{r^2} \\ r^2 &= \frac{6.67 \times 10^{-11} \times 1.99 \times 10^{30} \times 6.42 \times 10^{23}}{1.64 \times 10^{21}} \\ \therefore r &= 2.28 \times 10^{11} \text{ m} \end{aligned}$$

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Marking Criteria

- Correctly substitutes values into formula (1 mark) and correctly calculates answer (1 mark)

Question 3 (5 marks)

(a) If the average radius of the Earth is 6371 km, and the mass of the Earth is 6.00×10^{24} kg, calculate the acceleration due to gravity on the Earth's surface. Hint: you already know the answer you should be expecting here. (2 marks)

(b) The average radius of the Sun is about 700,000 km, and its mass is 1.99×10^{30} kg. Calculate the acceleration due to gravity on the Sun's surface. (2 marks)

(c) By what factor would your weight increase if you were to move from the surface of the Earth to the surface of the Sun? (1 mark)

Solution to Question 3

(a)

$$\begin{aligned} g &= \frac{GM}{r^2} \\ &= \frac{6.67 \times 10^{-11} \times 6.00 \times 10^{24}}{6371000^2} \\ &= 9.86\ \mathrm{m\,s^{-2}}\ \text{downwards} \end{aligned}$$

(b)

$$\begin{aligned} g &= \frac{GM}{r^2} \\ &= \frac{6.67 \times 10^{-11} \times 1.99 \times 10^{30}}{700000000^2} \\ &= 270.88\ \mathrm{m\,s^{-2}}\ \text{downwards} \end{aligned}$$

(c) Because mass remains constant, only the value of g affects weight force. Since the Sun's gravitational acceleration is $\frac{270.88}{9.86} = 27.47$ times greater than that of the Earth, you would weigh 27.47 times more on the Sun than on the Earth.

Marking Criteria

(a) Correctly substitutes values into formula (1 mark) and correctly calculates answer (1 mark)

(b) Correctly substitutes values into formula (1 mark) and correctly calculates answer (1 mark)

(c) Correctly determines ratio of accelerations (1 mark)

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Gravitational Fields

Suppose we had an object exerting one of the fundamental forces of nature, such as the electromagnetic or gravitational forces. There would be an area of influence around this object where other objects would be subject to its force. This area of influence is known as the force's **field**. For example, a proton has an electric field around it. If another charged object is in the field, the proton exerts a force on it. Similarly, all masses have an associated gravitational field, which influences every other object in its field by the force of gravity. A field is technically infinitely large, and continuous in space, but if we want to draw one, we obviously can't fulfil either of those criteria. So instead, we represent the field using **field lines**.

In the Preliminary course, we learned how to draw electric and magnetic field lines around electric monopoles, electric and magnetic dipoles, and electric plates. The direction of a field line represents the direction that a test particle would move in if placed at that point in the field. For electric fields, such a test particle is a tiny positive charge. For gravitational fields, the test particle is a tiny mass.

Recall that field lines are represented according to a set of conventions:

1. A stronger region of the field is indicated by field lines drawn closer together.
2. Because an object cannot move in two different directions at once, two field lines can never cross each other.
3. Field lines should emanate at 90° to the surface of the object creating the field, and should not penetrate into the object's interior.

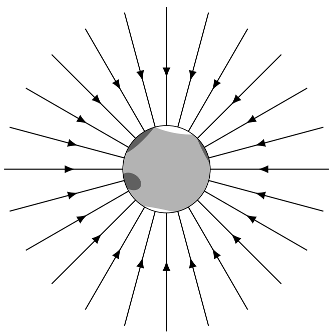

Figure 2 The Earth's gravitational field. Note that the field lines of a gravitational field always point inward.

Because the gravitational force is always attractive, unlike the electromagnetic force, field lines will always go inward towards the object creating the field: a test mass will always experience an attractive force and move towards the object.

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Factors Influencing Gravitational Acceleration

We calculated in one of the previous examples that the gravitational acceleration of an object on the Earth's surface is roughly 9.80 $m\,s^{-2}$, and that this value is independent of what object we are looking at, because the mass of the small object doesn't figure into the equation for g.

However, the value of g is actually not the same at all points on the surface of the Earth. For example, if you were to travel from Alaska to Sri Lanka, you would weigh 0.5% less stepping out of the plane than when you got on board. It is not that your mass has changed; it is just that the Earth is exerting a weaker force on you in Sri Lanka than it is in Alaska, and as such the acceleration you feel is lower. For a 100 kg person, this makes them feel half a kilogram lighter – that's not insignificant!

The main factors affecting acceleration can be summarised as follows:

Ellipsoidal shape of the Earth The Earth is not a perfect sphere. It is flatter at the poles and wider at the equator. The rough major axis radius of the at the equator is 6378 km and its rough minor axis radius at the poles is 6357 km. This 21 km difference leads to a small but significant difference in g when substituting these radii into the equation for gravitational acceleration - about am 0.7% difference. As a general rule, the closer your latitude is to the equator, the further you are from the centre of the Earth, and this slightly reduces g.

Heterogeneous composition of the Earth Remember that when we calculate g, we make the assumption that all of the Earth's mass is concentrated into its centre, which is its centre of mass. Although this is fine when looking at large scale interactions of gravity, such as between the Earth and the Sun, this isn't accurate when making calculations at the surface, and it is a complex calculation to see how each small part of the Earth leads to its own gravitational effect which all sum together. If you are on a part of the Earth's surface with dense crust underneath, such as oceanic crust, there is simply more mass in the line between your centre and the Earth's centre, than if you were in a region where the crust beneath you was less dense, like continental crust. This leads to differences in the gravitational acceleration you feel in these two regions.

Irregular surface altitudes The Earth is not smooth. There are regions on its surface which are much below sea level, such as the Mariana trench at almost 11 km deep, and there are regions much above sea level, such as Mt Everest at almost 9 km high. At these regions, the value for r in the calculation for g is different, and so the resultant value of g is different. At the top of a mountain, you are a small amount further away from the Earth's centre than you would be at sea level, so there is slightly less gravitational pull. But if you were at the bottom of

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the Mariana trench, you would be closer to the centre and the pull would be stronger.

Centrifugal effect of the Earth's rotation The Earth rotates about its central axis. Because all parts of the Earth need to complete one full rotation together in one day, it follows that the parts of the Earth further out from the central axis of rotation have to travel faster, becasue they need to cover a much longer circular path, than areas closer to the axis of rotation. This means that at the equator, the Earth is spinning quite fast – about 464 $\mathrm{m\,s^{-1}}$ – whereas at the North and South Poles, the Earth is essentially not spinning at all. When you are standing on the Earth's surface, just like when you are thrown outwards by inertia when a car turns fast, you are actually being thrown outwards tangentially to the Earth's surface, and this effect is greater where the Earth spins faster. Because the weight force is still much stronger than this effect, you don't actually fly outwards as the Earth spins, but it does detract from the *apparent* acceleration due to gravity you feel at the equator versus what you feel at the poles. It is important to remember that this factor *does not* influence the calculation of g like the previous three factors, but it influences the *apparent effect* of g.

So if your goal in life is to break the shot put world record, consider moving to Sri Lanka – all your projectiles will travel just that little bit further.

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Questions

Question 4 (6 marks)

The summit of Mt Chimborazo in Ecuador is the point on the Earth's surface furthest away from the Earth's centre, at 6384 km. Mt Chimborazo itself rises 6267 m above sea level.

(a) At 8848 m above sea level, Mt Everest is taller than Mt Chimborazo. Provide a reason why its summit is not as far away from the Earth's centre as the summit of Mt Chimborazo. (1 mark)

(b) Calculate the acceleration due to gravity of the Earth on the summit of Mt Chimborazo. Assume the mass of the Earth is 5.97×10^{24} kg. (2 marks)

(c) The acceleration due to the Earth's gravity in Oslo, Norway, is $9.825\ \mathrm{m\,s^{-2}}$. How much further would a projectile travel horizontally if it were thrown from the summit of Mt Chimborazo than if it were thrown from Oslo? Assume the projectile is launched from the ground and lands on the ground, and is launched at $10\ \mathrm{m\,s^{-1}}$ at 45° to the horizontal. (3 marks)

Solution to Question 4

(a) Mt Chimborazo is very close to the equator, which is where sea level is furthest away from the centre of the Earth. Mt Everest is further from the equator, so even though it is 2 km higher from sea level, it is actually closer to the centre of the Earth.

(b)
$$\begin{aligned} g &= \frac{GM}{r^2} \\ &= \frac{6.67 \times 10^{-11} \times 5.97 \times 10^{24}}{6384000^2} \\ &= 9.77\ \mathrm{m\,s^{-2}}\text{ downwards} \end{aligned}$$

(c) Both projectiles have the same velocity components, $u_x = 10\cos 45° = 7.07$ $\mathrm{m\,s^{-1}}$ and $u_y = 10\sin 45° = 7.07\ \mathrm{m\,s^{-1}}$. For the time of flight at Mt Chimborazo:

$$\begin{aligned} s &= ut + \frac{1}{2}at^2 \\ 0 &= 7.07t + \frac{-9.77t^2}{2} \\ t &= \frac{2 \times 7.07}{9.77},\ (t \neq 0) \\ t &= 1.447\ldots\ \mathrm{s} \end{aligned}$$

For the range at Mt Chimborazo:

$$
\begin{aligned}
s_x &= u_x t \\
&= 7.07 \times 1.447\ldots \\
&= 10.23 \text{ m}
\end{aligned}
$$

For the time of flight at Oslo:

$$
\begin{aligned}
s &= ut + \frac{1}{2}at^2 \\
0 &= 7.07t + \frac{-9.825t^2}{2} \\
t &= \frac{2 \times 7.07}{9.825}, \ (t \neq 0) \\
&= 1.439\ldots \text{ s}
\end{aligned}
$$

Thus, for the range at Oslo:

$$
\begin{aligned}
s_x &= u_x t \\
&= 7.07 \times 1.439\ldots \\
&= 10.18 \text{ m}
\end{aligned}
$$

Therefore, the projectile would travel 5 cm further thrown on the summit of Mt Chimborazo.

Marking Criteria

(a) Recognises Mt Chimborazo must be closer to the equator and bulges out further than Mt Everest (1 mark)

(b) Correctly substitutes values (1 mark) and calculates correct answer (1 mark)

(c) Calculates range for each location (1 mark each) and then finds the difference as the final answer (1 mark)

Question 5 (1 marks)

If the Sun and Earth exert the same magnitude of gravitational force on each other (by Newton's Third Law), why is the Earth's motion influenced to a far greater extent by the Sun than the Sun's motion is influenced by the Earth?

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Solution to Question 5

Although the Earth and Sun experience the same force, the Earth is so much lighter that its acceleration is significantly more. Since acceleration determines further kinematics, the motion of the Earth is much more affected than the motion of the Sun.

Marking Criteria

- Recognises that acceleration is different although forces are the same (1 mark)

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Fundamental Concepts

Newton's Law of Universal Gravitation Every object with mass in the universe attracts every other object with mass by a gravitational force, proportional to each mass and inversely proportional to the square of the distance between their centres of mass. The constant of proportionality is G, the gravitational constant. The formula is given by:

$$F = \frac{Gm_1m_2}{r^2}$$

Gravitational acceleration Derived by equating Newton's Law of Universal Gravitation with Newton's Second Law of Motion, and solving for acceleration. The formula is given by:

$$g = \frac{GM}{r^2}$$

Notably, the acceleration due to gravity exerted by object 1 on object 2 is independent of the mass of object 2. This is why when objects fall to Earth, they all experience the same gravitational acceleration independent of their mass.

Gravitational field A region of influence around a mass in which another mass will experience its force of gravity. Gravitational fields are represented by **field lines**, which are subject to the standard rules of field line representation, in that they cannot cross, they are perpendicular to the object's surface, and they do not extend into the object's interior. Gravitational field lines always point inwards, because the force of gravity is always attractive.

Factors influencing the value of g The Earth's surface doesn't have the same acceleration due to gravity at all points due to a number of factors. These factors include the ellipsoid shape of the Earth, the heterogeneity of the composition of the Earth, the different altitudes of the Earth's surface due to its geological features, and finally the variable centrifugal inertial effect of the Earth's rotation.

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2 Orbital Motion

Because of how massive both bodies are, the Sun attracts the Earth with a very strong force of gravity. But we know that the Earth isn't careening in towards the Sun in response to this force. Instead, the Earth travels around the Sun in a roughly circular (technically elliptical) path. This occurs because the Earth is not stationary, it has a velocity through space of about 30 $\mathrm{km\,s^{-1}}$. From the previous lessons, we know that when a moving object experiences a force which always points to a common centre, such as the gravitational force from the Sun, the resultant motion is circular, because a centripetal force is now in play.

The circular path of a moving object in space under the influence of the gravitational force of a central body is known as an **orbit**.

As we will see, we can bring in all our equations from circular motion and use them to describe orbital motion. We can start with one key observation: the centripetal force causing an object to orbit around a celestial body is solely comprised by the gravitational attraction. Therefore:

$$F_c = F_g$$

$$\frac{mv^2}{r} = \frac{GmM}{r^2}$$

Note that we are making the assumption here that all orbits are circular (Figure 3). In fact, all orbits in space are elliptical, with the central mass being orbited located at one of the elliptical foci, but the equations of circular motion are still quite good at describing orbital mechanics as long as the orbit isn't too **eccentric** (or non-circular).

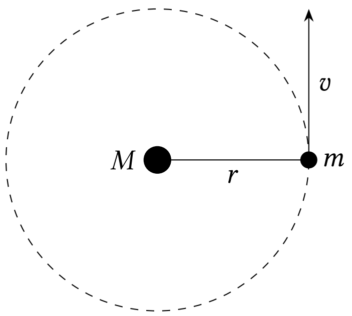

Figure 3 The motion of a small object orbiting a large central mass can be thought of as uniform circular motion if the orbit is assumed to be circular.

Before we continue, we should define the word **satellite**, which comes up often in this context. Satellites are bodies that orbits another body, and comprise **natural satellites** as well as **artificial satellites**. Natural satellites are bodies such as asteroids and moons. Planets are also satellites of the star they orbit. An artificial satellite is a human-made object that has been placed in orbit around any body. There are quite a

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few artificial satellites currently orbiting the Sun, and as we know, there are thousands orbiting the Earth, in addition to our lonely single natural satellite.

Orbital Acceleration

The orbital acceleration experienced by a satellite is equal to its centripetal acceleration, which is also equal to the central body's acceleration due to gravity g. Therefore, we can write:

$$a_{\text{orb}} = a_c = g$$
$$a_{\text{orb}} = \frac{v^2}{r} = \frac{GM}{r^2}$$

Orbital Velocity

From the orbital acceleration equation above, we can derive a formula for v, the tangential velocity of the orbiting body, in terms of the mass of the central body and the radius of the orbit:

$$\frac{v_{\text{orb}}^2}{r} = \frac{GM}{r^2}$$
$$v_{\text{orb}} = \sqrt{\frac{GM}{r}}$$

Thus, we can see that the velocity of an object's orbit only depends on the body it is orbiting and how far away it is from that body.

Orbital Period

Recall that the period of circular motion is the time taken to complete a full revolution. Thus, an orbital period is the time taken for an orbiting body to perform one full orbit around the central body. For planets, this is known as a **year**. Just as Earth's year is 365 days, a 'Mars year' – the time it takes the planet Mars to complete one revolution around the Sun – is 687 Earth days.

From circular motion, we know we can define the orbital period by dividing the orbital path length by the speed of the object:

$$T = \frac{2\pi r}{v}$$

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By substituting in the formula we derived for orbital velocity, we can derive an expression for period again purely based on the central body's mass and distance:

$$T = \frac{2\pi r}{\sqrt{\frac{GM}{r}}}$$
$$T = \frac{2\pi r^{\frac{3}{2}}}{\sqrt{GM}}$$

Orbital Radius

The orbital radius is the distance between the centres of mass of the orbiting body and the central body. It is the distance over which the gravitational force of each body is exerted on the other. This corresponds to the radius of the path of rotation from our basic circular motion analysis from previous weeks.

In most questions you encounter, the orbital radius will be given to you in the question stem. Alternatively, other questions will give you quantities such as the orbital velocity and the central mass, and ask you to find orbital radius. In this case, you can just rearrange the orbital velocity formula and solve for r. As such, there isn't really a standard formula for orbital radius. One such formula could be a rearrangement of the orbital velocity formula:

$$r = \frac{GM}{v^2}$$

Another such formula results from rearranging the formula for orbital period:

$$T = \frac{2\pi r^{\frac{3}{2}}}{\sqrt{GM}}$$
$$T^2 = \frac{4\pi^2 r^3}{GM}$$
$$r^3 = \frac{GMT^2}{4\pi^2}$$
$$\frac{r^3}{T^2} = \frac{GM}{4\pi^2}$$

This is one of Kepler's Laws of Planetary Motion, which we will examine in the next section.

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Questions

Question 6 (5 marks)

The Moon orbits the Earth at an average distance of 385,000 km. The mass of the Earth is 6.00×10^{24} kg.

(a) Calculate the Moon's orbital velocity. (1 mark)

(b) Calculate how long it takes the Moon to orbit the Earth, in Earth days. (2 marks)

(c) A 'Moon day', or the time taken for the Moon to complete one rotation about its own axis, is 27.46 Earth days long. Using your answer from part (b), what does the length of the Moon day imply about the visibility of the Moon's surface from Earth? (2 marks)

Solution to Question 6

(a)

$$\begin{aligned} v &= \sqrt{\frac{GM}{r}} \\ &= \sqrt{\frac{6.67 \times 10^{-11} \times 6.00 \times 10^{24}}{385000000}} \\ &= 1019.5\ \mathrm{m\,s^{-1}} \end{aligned}$$

(b)

$$\begin{aligned} T &= \frac{2\pi r}{v} \\ &= \frac{2\pi \times 385000000}{1019.5} \\ &= 2372757.57\ldots\ \mathrm{s} \end{aligned}$$

Dividing by $24 \times 60 \times 60$ seconds in a day, we get 27.46 Earth days.

(c) We have just calculated that the Moon takes the same amount of time to orbit the Earth that it does to rotate on its own axis. This means that as the Moon orbits, it turns in a way that we always see the same side of it. There is a far side of the Moon which we never see from Earth because it is always rotating away from us.

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Marking Criteria

(a) Calculates velocity correctly (1 mark)

(b) Correct calculation of period in seconds (1 mark), then converts this to a number of days (1 mark).

(c) Identifies that the Moon's orbit and Moon's rotation take the same amount of time (1 mark), and recognises this means the same side of the Moon always faces the Earth during its orbit (1 mark)

Question 7 (3 marks)

The Earth takes 365.25 days to orbit the Sun, orbiting at a distance of 1 astronomical unit, or 149,800,000 km. Calculate:

(a) The Earth's orbital velocity. (1 mark)

(b) The mass of the Sun. (1 mark)

(c) The acceleration due to the Sun's gravity as felt by the Earth. (1 mark)

Solution to Question 7

(a)

$$v = \frac{2\pi r}{T}$$
$$v = \frac{2\pi \times 149800000000}{365.25 \times 24 \times 60 \times 60}$$
$$v = 29825.5\ \mathrm{m\,s^{-1}}$$

(b)

$$v = \sqrt{\frac{GM}{r}}$$
$$M = \frac{v^2 r}{G}$$
$$= \frac{29825.5^2 \times 149800000000}{6.67 \times 10^{-11}}$$
$$= 1.998 \times 10^{30}\ \mathrm{kg}$$

(c)

$$g = \frac{GM}{r^2}$$
$$= \frac{6.67 \times 10^{-11} \times 1.998 \times 10^{30}}{149800000000^2}$$
$$= 5.94 \times 10^{-3}\ \mathrm{m\,s^{-2}} \text{ towards the Sun}$$

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Marking Criteria

(a) Calculates answer correctly (1 mark)

(b) Calculates answer correctly (1 mark)

(c) Calculates answer correctly (1 mark)

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Kepler's Laws of Planetary Motion

Johannes Kepler (1571–1630) was a German astronomer and an assistant of Danish astronomer Tycho Brahe, one of the great observational scientists of history. Kepler studied the voluminous and meticulous data recorded by Brahe during his life, and formulated three laws of planetary motion which we still use today. Kepler lived before Newton, so these laws were purely based on observational data and were not based on any sort of physical theory of gravity. In fact, Kepler's laws proved to be a great influence on Newton's formulation of his theory of Universal Gravitation. Although named for the motion of planets, Kepler's laws hold true for all orbiting bodies.

Kepler's three Laws of Planetary Motion are as follows:

1. All planets undertake elliptical orbits, with the central body (star) at one of the foci of the ellipse.

2. The sectors subtended by arcs of a particular planet's orbit, traversed over equal periods of time, will have equal areas.

3. For all planets orbiting the same central mass M, the cube of the (major) radius of orbit is proportional to the square of the period of orbit, given by the formula $\frac{r^3}{T^2} = \frac{GM}{4\pi^2}$.

Kepler's First Law reflects a fundamental observation of orbits in the universe. Some orbits are very elliptical (high eccentricity), whereas others, such as the orbit of the Earth, are almost circular (eccentricity near zero). Pluto's orbit is quite eccentric, which was one of the reasons it was removed from the list of planets in our Solar System. This means that Pluto's orbit is a very deformed ellipse, so its distance from the Sun at the point where it is closest in its orbit (perihelion) and the distance from the Sun where it is furthest (aphelion) are very different lengths. By contrast, the perihelion and aphelion of the Earth are very similar in length. That being said, the Earth still doesn't have a perfectly circular orbit. It is actually closer to the Sun at the time of the year where the Southern Hemisphere experiences summer, and is slightly further away from the Sun during the Northern Hemisphere's summer. This makes no appreciable difference to temperatures experienced during those seasons, however.

Kepler's Second Law is a consequence of the relationship between orbital radius and velocity, $v = \sqrt{\frac{GM}{r}}$. When the orbiting body is further away in its orbit from the central mass, it travels slower. In other words, although its distance from the central mass is greater, it does not traverse as great an arc in a given amount of time because it moves slower. As such, the sector it subtends with an arc of its motion is thin and long. When the orbiting mass is closer to the central mass, it travels faster, so in the same amount of time it traverses a longer arc. However, the radius of its orbit is lower,

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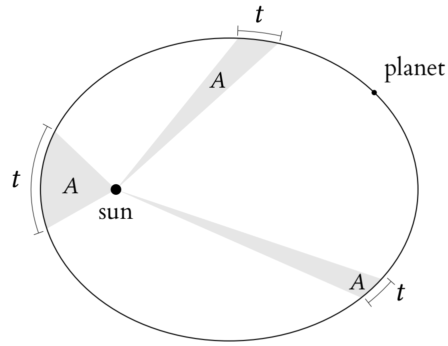

Figure 4 Kepler's Second Law states that an object in an elliptical orbit will sweep out sectors of equal area, subtended at the central body, over equal periods of time.

so the sectors it sweeps out are shorter but thicker. The areas of each sector end up being the same.

As for Kepler's Third Law, we have already derived it earlier by rearranging the formula for orbital period. An easier way to derive this law, however, is to rearrange the basic formula for orbital period with v as the subject:

$$T = \frac{2\pi r}{v}$$
$$v = \frac{2\pi r}{T}$$

Then we substitute this into the formula for orbital velocity:

$$v = \sqrt{\frac{GM}{r}}$$
$$\frac{2\pi r}{T} = \sqrt{\frac{GM}{r}}$$
$$\frac{4\pi^2 r^2}{T^2} = \frac{GM}{r}$$
$$\frac{r^3}{T^2} = \frac{GM}{4\pi^2}$$

Energy and Kepler's Laws

Kepler's Laws of Planetary Motion can be investigated from the perspective of the total energy in orbiting systems. To look at this in more detail, we need to look at a new definition of gravitational potential energy, and the mechanical energy of satellites. We will do this in the next lesson, and therefore revisit Kepler's Laws in relation to energy at that time.

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Fundamental Concepts

Orbit The closed-loop path taken by an object moving in space subject to a gravitational force exerted by a large central mass. Orbits can be approximated as circular, and as such we can rely on our understanding of circular motion and use those formulas in conjunction with our gravitational formulas to describe the orbital behaviour of objects in the universe.

Satellite A natural satellite, which is any astronomical body orbiting another astronomical body, or an artificial satellite, which are human-made objects placed in orbit around a variety of objects in our Solar System.

Centripetal force in orbits The centripetal force keeping an object in orbit is the force of gravity, given by Newton's law of universal gravitation $F = \frac{GmM}{r^2}$.

Centripetal acceleration of orbiting bodies This is the acceleration due to gravity from the central object, given by $g = \frac{GM}{r^2}$.

Orbital velocity The orbital velocity of an object undergoing an orbit at a distance r from a central body of mass M is given by $v = \sqrt{\frac{GM}{r}}$.

Orbital period The orbital period is given by $T = \frac{2\pi r}{v}$. We can take this one step further by substituting the formula for orbital velocity to obtain $T = \frac{2\pi r^{\frac{3}{2}}}{\sqrt{GM}}$. It is not necessary to memorise this second equation at all – you could simply work up to it if a question in an exam required it of you.

Orbital radius The distance between the satellite's centre of mass and the centre of mass of the body it is orbiting.

Kepler's Laws of Planetary Motion Kepler's laws can be used to explain and make predictions about the orbits not only of planets, but also of all satellites in the universe. Kepler's three laws are as follows:

1. All planets undertake elliptical orbits, with the central body (star) at one of the foci of the ellipse.
2. The sectors subtended by arcs of a particular planet's orbit, traversed over equal periods of time, will have equal areas.
3. For all planets orbiting the same central mass M, the cube of the (major) radius of orbit is proportional to the square of the period of orbit, given by the formula $\frac{r^3}{T^2} = \frac{GM}{4\pi^2}$.

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Questions

Question 8 (3 marks)

A satellite is currently orbiting the Earth at an altitude above the surface of 400 km, and completes one full orbit every 90 minutes. The satellite then fires its thrusters to move up to an orbital altitude of 1000 km. Given that the Earth has an average radius of 6371 km, what is the satellite's new period of rotation in hours?

Solution to Question 8

We need to remember here that orbital altitude is *not* orbital radius, we need to add the radius of the Earth each time. Given that at each orbital radius, the satellite has the same value of $\frac{GM}{4\pi^2}$ because it is still orbiting the Earth, we can just set up Kepler's Third Law as a set of two ratios which are equal. Furthermore, we can just use km and hours because we are comparing ratios and not using any standard constants:

$$\begin{aligned}
\frac{r_{old}^3}{T_{old^2}} &= \frac{r_{new}^3}{T_{new}^2} \\
\frac{6771^3}{1.5^2} &= \frac{7371^3}{T_{new}^2} \\
T_{new}^2 &= 2.9027\ldots \\
T_{new} &= 1.70 \text{ hours}
\end{aligned}$$

Marking Criteria

- Sets up Kepler's Third Law as ratios (1 mark) Note: The student could use the initial period and radius to solve for the Earth's mass and do this without ratios, but it is longer. The student is not allowed to assume the mass of the Earth.
- Correctly substitutes values (1 mark)
- Calculates final answer correctly (1 mark)

Question 9 (8 marks)

The four Galilean moons of Jupiter are Io, Europa, Ganymede and Callisto. Their discovery was instrumental in disproving the geocentric model of the universe. Io, Europa and Ganymede have their orbital periods in a ratio of 1 : 2 : 4.

(a) What is the ratio of the orbital radii of Io, Europa and Ganymede in that order? (2 marks)

(b) If Europa's orbital radius is 670,000 km, calculate the orbital radii of Io and Ganymede in kilometres. (2 marks)

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(c) If Io's orbital period is 1.77 Earth days, calculate the mass of Jupiter. (Assume one Earth day = 24 hours.) (2 marks)

(d) If Callisto's orbital radius is 1,833,000 km on average, calculate the its orbital period in Earth days. (2 marks)

Solution to Question 9

(a) Because all moons orbit the same central mass (Jupiter), $\frac{GM}{4\pi^2}$ is identical for all of them. Therefore, they all have the same $\frac{r^3}{T^2}$ value in their orbital paths.

$$\frac{r_I^3}{T_I^2} = \frac{r_E^3}{T_E^2} = \frac{r_G^3}{T_G^2}$$
$$\frac{r_I^3}{1} = \frac{r_E^3}{4} = \frac{r_G^3}{16}$$
$$\frac{r_I}{1} = \frac{r_E}{1.587} = \frac{r_G}{2.520}$$

Therefore their orbital radii are in the ratio 1 : 1.587 : 2.520.

(b) Using the ratios above:

$$r_I = \frac{670000}{1.587} = 422180 \text{ km}$$
$$\text{and } r_G = 670000 \times \frac{2.520}{1.587} = 1063894 \text{ km}$$

(c)
$$\frac{r^3}{T^2} = \frac{GM}{4\pi^2}$$
$$M = \frac{4\pi^2 r^3}{GT^2}$$
$$= \frac{4\pi^2 \times 422180000^3}{6.67 \times 10^{-11} \times (1.77 \times 24 \times 60 \times 60)^2}$$
$$= 1.90 \times 10^{27} \text{ kg}$$

(d)
$$\frac{r^3}{T^2} = \frac{GM}{4\pi^2}$$
$$T^2 = \frac{4\pi^2 r^3}{GM}$$
$$T^2 = \frac{4\pi^2 \times 1833000000^3}{6.67 \times 10^{-11} \times 1.90 \times 10^{27}}$$
$$T^2 = 1.9185\ldots \times 10^{12}$$
$$\therefore T = 1385108.672\ldots \text{ s}$$

Dividing by $24 \times 60 \times 60$ seconds in an Earth day, we get a period of 16.03 Earth days.

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Marking Criteria

(a) Sets up Kepler's Third Law as ratios (1 mark), calculates correct radii ratios (1 mark)

(b) Calculates radius of Io and Ganymede correctly (1 mark each)

(c) Substitutes correctly into formula (1 mark), calculates correct mass (1 mark)

(d) Calculates correct period in seconds (1 mark), calculates correct period in days (1 mark)

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Quiz Solutions

Question 1

(a)
$$\begin{aligned} F_c &= \frac{mv^2}{r} \\ &= \frac{880 \times (\frac{190}{3.6})^2}{850} \\ &= 2883.8 \text{ N} \end{aligned}$$

(b) The 30° inward turn creates a component of friction in line with the car's motion and a component perpendicular to it. We know the perpendicular component is 2883.8 N. Therefore, the total friction is $\frac{2883.8}{\sin 30°} = 5767.6$ N.

(c) Angular frequency:
$$\begin{aligned} \omega &= \frac{v}{r} \\ &= \frac{\frac{190}{3.6}}{850} \\ &= 0.062 \text{ s}^{-1} \end{aligned}$$

For period, we can use $\omega = \frac{2\pi}{T}$:
$$\begin{aligned} T &= \frac{2\pi}{0.062} \\ &= 101.2 \text{ s} \end{aligned}$$

(d) When a tight turn is made quickly, v is large and r is small. This makes $F_c = \frac{mv^2}{r}$ very large, and this is all supplied by friction between the tyres and the road. Because wheels undergo static friction with the road at their point of contact when rotating, and static friction is given by $F_f <= \mu_s N$, there is a limit of friction force at which static friction is exceeded. After this, the wheel experiences kinetic friction, and starts *moving* across the road instead of rotating, i.e. sliding in the direction the car is travelling, which is tangential. As the centripetal effect of friction at this point is lost, the car skids in a tangent due to its inertia, which acts centrifugally, and travels outwards from the track.

Question 2

We can use the maximum height to find the initial vertical velocity. The maximum height is a displacement of 12 m, not 32 m.

$$\begin{aligned} v^2 &= u^2 + 2as \\ 0^2 &= u_y^2 + 2 \times -9.80 \times 12 \\ u_y &= 15.34 \text{ m s}^{-1} \end{aligned}$$

This means that $u_x = 15.34\ \mathrm{m\,s^{-1}}$ too. Now we can use u_y to find the time of flight:

$$\begin{aligned} s &= ut + \frac{1}{2}at^2 \\ -20 &= 15.34 \times t - 4.9 \times t^2 \\ 4.9t^2 - 15.34t - 20 &= 0 \end{aligned}$$

Using the quadratic equation and taking $t > 0$, we get:

$$\begin{aligned} t &= \frac{15.34 + \sqrt{15.34^2 - 4 \times 4.9 \times -20}}{9.8} \\ &= 4.12\ \mathrm{s} \end{aligned}$$

Now we can find the range:

$$\begin{aligned} s_x &= u_x t \\ &= 15.34 \times 4.12 \\ &= 63.2\ \mathrm{m} \end{aligned}$$

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